Lecture 10

$$\underset{\Phi}{\overset{\text{max abelian}}{\mathfrak{h}}} \subset \underset{\Phi^+}{\overset{\text{max solvable}}{\mathfrak{b}}} \subset \mathfrak{g} \quad \text{Cartan \& Borel in semisimple } \mathfrak{g}$$

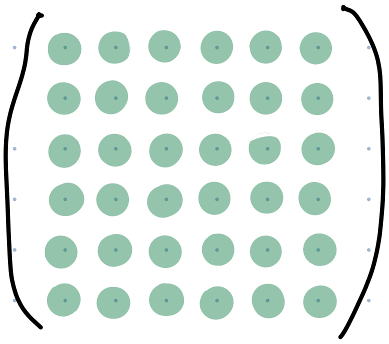
Φ^+ positive roots ($\Phi^- = -\Phi^+$)

Δ simple roots

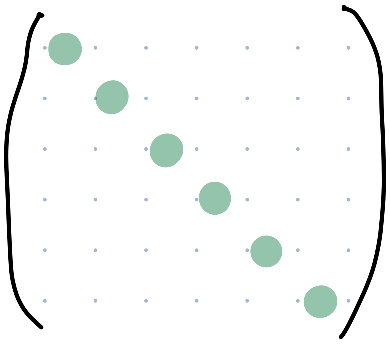
For $S \subset \Delta$, let $\Phi_S^- = (\mathbb{Z}\text{-span } S) \cap \Phi^-$, $\mathfrak{p}_S = \mathfrak{b} \oplus \bigoplus_{\alpha \in \Phi_S^-} \mathfrak{g}_\alpha$

So $S = \phi$ means $\mathfrak{p}_\phi = \mathfrak{b}$, $S = \Delta$ means $\mathfrak{p}_S = \mathfrak{g}$.

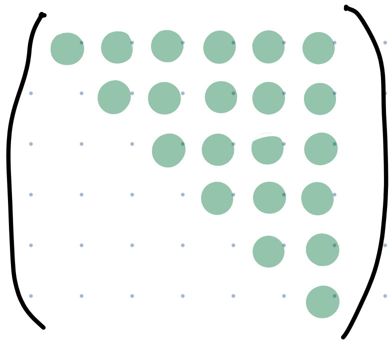
<u>Case A_n</u> ($\mathfrak{sl}_{n+1}\mathbb{C}$) $n=5$



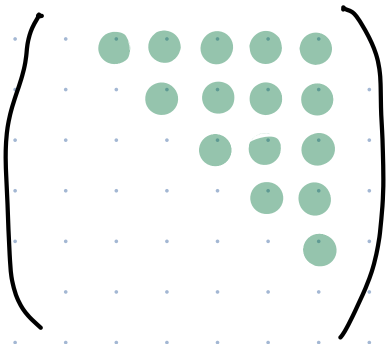
$\mathfrak{g}$



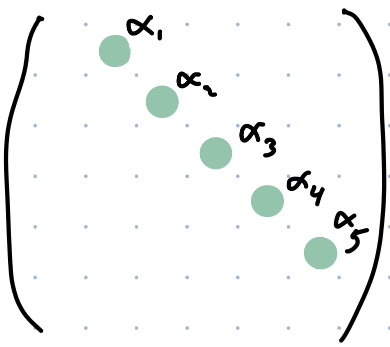
$\mathfrak{h}$



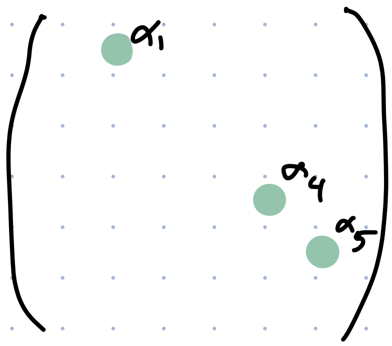
$\mathfrak{b}$



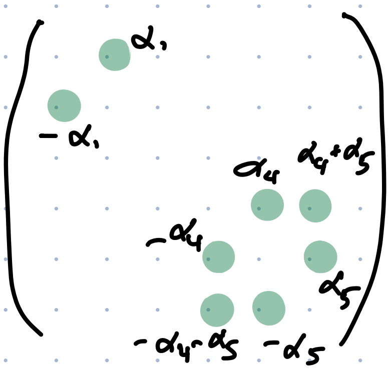
Φ^+



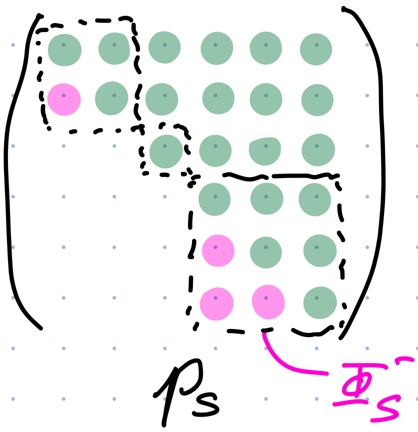
Δ



$S \subset \Delta$



$\mathbb{Z}\text{-span } S \cap \Phi$



$\mathfrak{p}_S$

$\cong \mathrm{Lie}(P_{\underline{d}})$ $\underline{d} = (2,1,3)$

The diagram of a parabolic subalg

$S \subset \Delta \rightsquigarrow$ Dynkin diag but simple roots in S marked $\bullet$, else $\times$

e.g. $\bullet\!-\!\!\times\!\!\Rightarrow\!\bullet$ means $\mathfrak{p}_S$, $S=\{\alpha,\gamma\}$ when $\Delta=\{\alpha,\beta,\gamma\}$ γ short.

From Lie alg to Lie Groups

Complex semisimple Lie group := Complex Lie group ($\mathbb{C}$ mfld, holo maps) s.t. G has finitely many connected cpnts and $\mathrm{Lie}(G)=\mathfrak{g}$ is a semisimple Lie alg.

To each subalgebra $\mathfrak{a} \subset \mathfrak{g}$ we have a Lie subgroup (immersed submfd) $A \subset G$ with $T_eA = \mathfrak{a} \subset \mathfrak{g} = T_eG$. If we ask A to be connected, then it is uniquely det by $\mathfrak{a}$.

But in $Flag(V) = \mathrm{Aut}(V)/P$ we needed P <u>closed</u>.

In general it is very difficult to say which $\mathfrak{a} \subset \mathfrak{g}$ corresp to closed (hence embedded Lie) subgroups.

<u>Normalizers</u>. $\mathfrak{a} \subset \mathfrak{g}$ subspace

$$n_{\mathfrak{g}}(\mathfrak{a}) = \{x \mid [x,\mathfrak{a}] \subset \mathfrak{a}\} = \{x \mid ad_x(\mathfrak{a}) \subset \mathfrak{a}\} \quad \text{Normalizer of } \mathfrak{a} \text{ in } \mathfrak{g}.$$

$$N_G(\mathfrak{a}) = \{g \mid Ad_g(\mathfrak{a}) = \mathfrak{a}\} \quad \text{where} \quad Ad_g(x) = d(C_g)_e(x)$$

Normalizer of $\mathfrak{a}$ in G. $\qquad C_g(h) = ghg^{-1}$.

<u>Note</u>: $Ad: G \to \mathrm{Aut}\,\mathfrak{g}$ has deriv $ad: \mathfrak{g} \to \mathrm{End}\,\mathfrak{g}$

<u>Note</u>: for matrix Lie alg, $Ad_A(X) = AXA^{-1}$.

<u>Thm</u>. $N_G(\mathfrak{a})$ is a closed subgroup of G with Lie algebra $n_{\mathfrak{g}}(\mathfrak{a})$.

<u>Pf</u>. Let $d = \dim \mathfrak{a}$. Through Ad_g G acts smoothly on $Gr(d,\mathfrak{g})$. Stabilizers in a smooth (or cts!) action are closed.

And $N_G(\mathfrak{a}) = \mathrm{Stab}(\mathfrak{a})$ where $\mathfrak{a}$ considered a point of $Gr(d, \mathfrak{g})$.

Now, $(d\,Ad)_e = ad$ means a path in $N_G(\mathfrak{a})$ has tangent vec at e that's in $n_{\mathfrak{g}}(\mathfrak{a})$.
Conversely the BCH formula shows that $\exp(tX) \in N_G(\mathfrak{a})$ if $X \in n_{\mathfrak{g}}(\mathfrak{a})$.

$\square$

<u>Thm</u>. $\mathfrak{b} = n_{\mathfrak{g}}(\mathfrak{b})$.
<u>Pf</u>. Follows from the statement (we didn't prove) that $\mathfrak{b}$ is an example of a maximal solvable subalg.
Suppose $x \in n_{\mathfrak{g}}(\mathfrak{b})$. Let $\mathfrak{b}'$ = algebra generated by $\mathfrak{b}$ and x.
If $x \notin \mathfrak{b}$ then $\mathfrak{b}' = \mathfrak{b} \oplus (\mathbb{C} \cdot x)$ because this is closed under Lie $[,]$.
But $[\mathfrak{b}', \mathfrak{b}'] \subset \mathfrak{b}$ because $[x, x] = 0$, $[x, \mathfrak{b}] \subset \mathfrak{b}$.
So $\mathfrak{b}'$ is a strictly larger maximal solvable subalg.
Contradiction! So $x \in \mathfrak{b}$. Thus $n_{\mathfrak{g}}(\mathfrak{b}) = \mathfrak{b}$.

$\square$

<u>Thm</u>. $\mathfrak{p}_S = n_{\mathfrak{g}}(\mathfrak{p}_S)$.
<u>Pf</u>. Follows from: $\mathfrak{b}$ is a max solvable subalg of $\mathfrak{p}_S$, and any two conj.
So if $x \in n_{\mathfrak{g}}(\mathfrak{p}_S)$ then $Ad_{e^{tx}}\mathfrak{b} \subset \mathfrak{p}_S$ is another max solvable in $\mathfrak{p}_S$.
Hence $\exists$ $Y \in \mathfrak{p}_S$ s.t. $Ad_{e^{Y_t}}\mathfrak{b} = Ad_{e^{tx}}\mathfrak{b}$.

$$\Rightarrow \mathfrak{b} = Ad_{e^{-Y_t}e^{tx}}\mathfrak{b} \Rightarrow e^{-Y_t}e^{tX} \in N_G(\mathfrak{b})$$
$$\Rightarrow X - \dot{Y} \in \mathfrak{b}$$
$$\Rightarrow X \in \mathfrak{b} + \mathfrak{p}_S = \mathfrak{p}_S.$$

$\square$

<u>Cor</u>. We have subgroups B, $P_S \subset G$ defined as normalizers of the corresp subalgebras.
and $P_\phi = B$.

Def. A parabolic subgroup of G is one of the form $N_G(\mathfrak{p})$ (or Lie $=\mathfrak{p}$) for $\mathfrak{p} \subset \mathfrak{g}$ a parabolic subalg ($\Leftrightarrow \mathfrak{p}$ conj to $\mathfrak{p}_S$).
$\Leftrightarrow \mathfrak{p}$ contains Borel.

Thm. $B = P_\emptyset$ is a Borel subgroup, i.e. maximal solvable connected.

Thm (Chevalley). Parabolic subgroups are connected and self-normalizing.

Def. A (partial) flag variety of G is a complex manifold of the form G/P where $P \subset G$ is parabolic subgroup.

G/B is the full flag variety.

Note G/H only depends on the conjugacy class of H, so can take $P = P_S$.

Next we'll work toward:

Compactness theorem: G/P is compact.

Thm. If G is a semisimple complex Lie group, then G is the underlying $\mathbb{C}$ mfld of a linear algebraic group. Closed $\mathbb{C}$ subgrps corresp to alg sub.

Pf. Onishchik-Vinberg ...

Thm. Every closed subgrp is the stab of a line in a linear rep.

Thm (Borel 1956**).** If a conn solvable lin alg grp acts alg on a complete variety (e.g. projective one), then it has a fixed pt.

Pf of cptness. Let $B = \mathrm{Stab}\, \ell$. ($\ell \subset V$) B has a fixed pt acting on $\mathrm{Flag}(V/\ell)$. Get $\ell \subset F_2 \subset \cdots \subset F_d$ flag of V, call it $F \in \mathrm{Flag}(V)$. Thus $\rho(G) \subset$ upper tri mats.

Then $B = \mathrm{Stab}_G(F)$, $G/B = G \cdot F$.

Any other stabilizer is solvable, as it lies in upper tri.
So $G \cdot F$ is an orbit of min dim. $\Rightarrow$ closed $\Rightarrow$ cpt

$\square$